Optimum Midcourse Plane Changes for Ballistic Interplanetary Trajectories

W. R. FIMPLE[1]

United Aircraft Corporation, East Hartford, Conn.

An analysis is presented of midcourse plane changes for ballistic interplanetary trajectories designed to eliminate certain recurring periods of high velocity requirements that are associated with interplanetary trajectories in the absence of plane changes. The midcourse plane change is optimized so as to minimize the required total velocity increment for both planetary probe missions and planetary satellite missions. Illustrating an application, curves of trip time and velocity requirement vs launch date are presented for Earth-Mars trajectories during the 1964 to 1965 time period. The velocity requirements for trajectories with midcourse plane changes are compared to the requirements for corresponding single-impulse trajectories that lie in a single plane.

Nomenclature

a = semi-major axis of two-dimensional transfer conic
i, j, k = unit orthogonal vector triad
K = as defined in Eqs. [B11–B16]
l = semilatus rectum of two-dimensional transfer conic
m = as defined in Eq. [A15]
n = order of expansion in Eq. [B9]
N = unit normal vector
q = as defined in Eq. [A15]
r = radius
r = radius vector
V = velocity vector
ΔV = velocity change
X = as defined in Eq. [B7]
Y = as defined in Eq. [B6]
Z = as defined for Eq. [A3]
γ = angle between periapsis and departure point D
θ = central angle
μ = gravitational constant
ν = circumferential component of velocity
ρ = radial component of velocity
ϕ = angle of inclination of orbital and trajectory planes
ψ = as defined in Eq. [A12]

Subscripts

1 = orbital plane of the initial planet
2 = first trajectory plane
3 = second trajectory plane
4 = orbital plane of the target planet
A = position of arrival at target planet
D = position of departure from initial planet
I = position of intermediate impulse application
N = position of ascending node
0 = refers to circular planetary satellite orbit
sta = indicates stationary value
∞ = refers to infinite distance from a planet

The subscripts D, I, and A used with ρ and ν refer to the radial and circumferential velocities of the two-dimensional transfer at $\theta = 0$, $\theta = \theta_I$, and $\theta = \theta_A$, respectively. The subscripts 1 and 4 used with ρ and ν refer to these same quantities in the initial and target orbits at points D and A, respectively.

Superscript

* = indicates base value for Taylor expansion

A NUMBER of studies have appeared in the recent literature which treat the problem of high-thrust ballistic interplanetary transfer in a realistic solar system, i.e., with inclined and elliptical planetary orbits. Among these are Refs. 1–3. Only single-impulse probes have been considered in these studies. With single-impulse trajectories, however, certain ranges of launch time and transfer time require intolerably large changes in velocity. Some of these high-velocity periods can be eliminated by means of a second midcourse impulse used to change the plane of the trajectory as originally proposed in Ref. 4 and also mentioned in Ref. 5. The object of this study was to determine the optimum position of the midcourse plane change for minimum velocity requirements and to present an example of the advantage of this maneuver.

Discussion of Problem

As illustrated in Fig. 1, the plane of a single-impulse, planetary-intercept trajectory, labeled 2 in the figure, is determined by three points: the initial planet at departure, point D; the sun; and the target planet at arrival, point A. Because of the inclinations of the planetary orbits, highly inclined trajectory planes and large changes in velocity are required for the ranges of launch time and trip time in which points D and A are in the neighborhood of opposition.

These high-velocity requirements for single-impulse, planetary-intercept trajectories in the vicinity of the 180° trajectory can be avoided by using a second intermediate impulse to change the trajectory plane. Details of this trajectory are shown in Fig. 2. Planes 1 and 4, the orbital planes of the initial and target planets, are inclined by an angle ϕ_4 and intersect in the line of nodes. The first impulse applied at point D directs the vehicle onto a ballistic trajectory in plane 2, inclined to plane 1 by the dihedral angle ϕ_2. Upon arrival at point I, a second impulse is applied normal to plane 1 to divert the trajectory onto plane 3. The trajectory in plane 3 intercepts the target planet at point A, where a third impulse may be applied if required to establish a satellite orbit.

Looking at the problem qualitatively, the optimum location of the intermediate impulse point depends upon two considerations. Upon initial inspection, it may appear that minimum ΔV corresponds to a minimum total change in the trajectory plane at points D and I. The second consideration, however, that the ΔV required to accomplish any given change in the trajectory plane increases with flight velocity, indicates that minimum ΔV occurs at a point on the trajectory where the velocity may be less than that at the point of minimum plane change. Closed form analytic solutions are presented in the Appendixes for the location of point I for minimum total velocity requirements for both the probe and the satellite.

To facilitate the analysis, a number of simplifying assumptions are made. The transfer is treated as a series of two-

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[1] Analytical Research Engineer, Research Laboratories. Member AIAA.

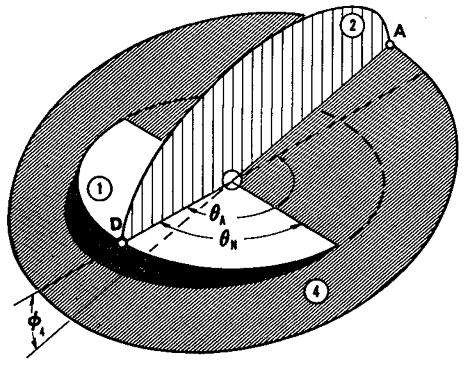

Fig. 1 One-impulse interplanetary trajectory

body problems: hyperbolic escape from the initial planet; transfer in heliocentric space; and capture or flyby in the vicinity of the target planet. Throughout the period of the transfer, the planetary orbits are assumed to be fixed ellipses. Small angle approximations are used. This is warranted by the structure of the solar system, where the inclinations of the planetary orbital planes are small except possibly for those of Mercury and Pluto.

Specifically, the approximations are made in the following manner. For a given launch date and trip time, a single-impulse, two-dimensional trajectory in plane 1 is found between point D and point A_1, the projection of the target planet on plane 1 at the time of arrival. A computing program taken from Ref. 3 is used to calculate these two-dimensional trajectories. Then, taking into account the inclinations of the planetary orbital planes, the three-dimensional trajectory is formed by the intersections of planes 2 and 3 with a cylindrical surface generated by an element perpendicular to plane 1 following the two-dimensional trajectory. These two intersections generally are not the correct conic sections required for ballistic motion in the central force field, but, because the angles involved are very small, they are acceptably close to the correct conic sections. This model greatly simplifies the mathematical analysis for the determination of minimum ΔV conditions and thereby allows analytic solutions to be obtained.

Results

Only calculations for trajectories from Earth to Mars are presented, since the objective is to provide some representative results of the analysis and not to present a complete set of data. An interesting range of launch dates extending from May 1964 to September 1965 was chosen because it contains favorable periods as well as a period containing 180° trajectories in which the very large ΔV requirements of the single-impulse case can be alleviated by using a second intermediate impulse.

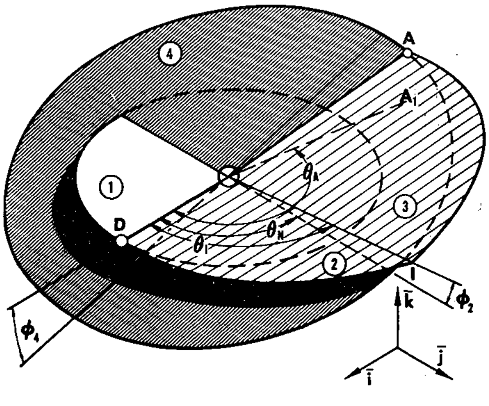

Fig. 2 Two-impulse interplanetary trajectory

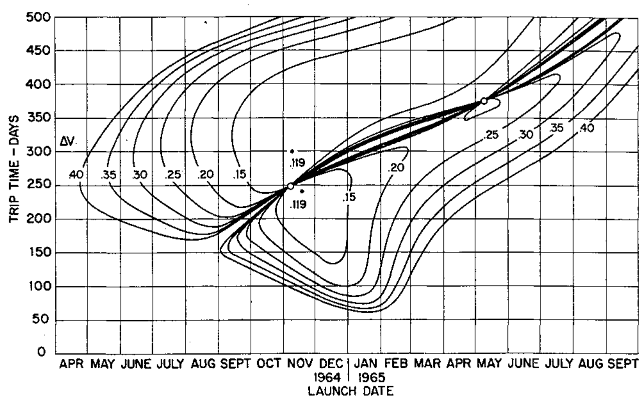

Fig. 3 Velocity requirements for single-impulse Mars probes; ΔV is normalized with respect to mean orbital speed of Earth

Compiled through the use of the Ref. 3 computer program, Fig. 3 shows constant ΔV contours for single-impulse ballistic probes to Mars plotted on a grid of trip time and launch date. This figure corresponds to similar figures presented in Ref. 2 except that the Ref. 2 curves are contours of constant V_∞, i.e., the hyperbolic excess velocity with respect to a planet. The ΔV presented here is the actual increase in velocity from that of a circular orbit around Earth at an altitude of 300 naut miles (24,880 fps). The values of ΔV have been normalized with respect to the mean orbital velocity of Earth (97,700 fps).

The two points in Fig. 3 through which most of the contours pass correspond to 180° nodal trajectories. The probe leaves Earth when it is crossing the nodal line and 180° later reaches Mars as it is crossing the opposite branch of the nodal line. For these two particular sets of trip time and launch date, the trajectory plane can be inclined at any angle to the ecliptic, since the points D, A, and the sun lie in a straight line. Between and on each side of these nodal transfer points runs a steep ridge of ΔV contours which separates two regions of relatively low velocity requirements. This is the region that is eliminated by the use of an intermediate impulse to change the trajectory plane.

Fig. 4 shows a cross section of Fig. 3 for a constant trip time of 300 days. Outside the period of high single-impulse velocity requirements, the single-impulse curve was obtained by two methods: 1) by exact calculations using the computing machine program of Ref. 3, and 2) by using the Ref. 3 machine program to find the two-dimensional trajectory for the approximate model previously described. That the two sets of calculations correspond exactly within plotting accuracy verifies the validity of the approximations. Inside the period of high single-impulse velocity requirements, the

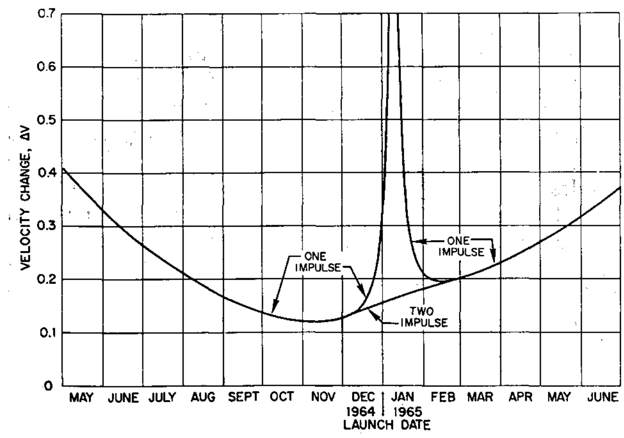

Fig. 4 Velocity requirements for one- and two-impulse Mars probes; trip time = 300 days; ΔV is normalized with respect to mean orbital speed of Earth

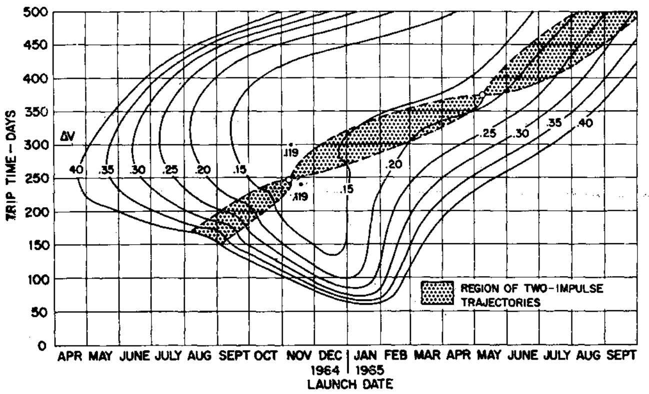

Fig. 5 Minimum velocity requirements for two-impulse Mars probes; ΔV is normalized with respect to mean orbital speed of Earth

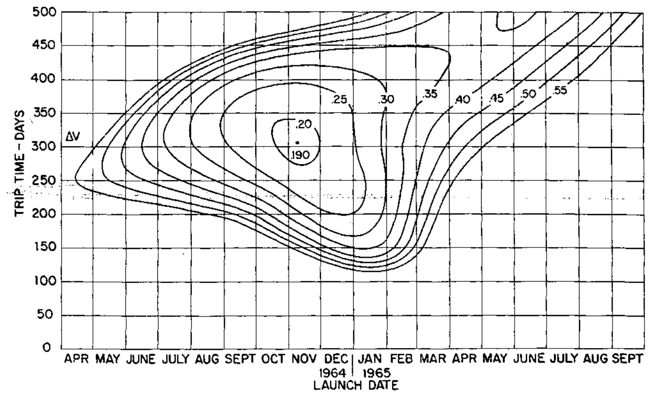

Fig. 6 Minimum velocity requirements for three-impulse Mars satellites; ΔV is normalized with respect to mean orbital speed of Earth

two-impulse trajectory is employed to eliminate the high peak of the single-impulse curve.

Fig. 5 is similar to Fig. 3 except that the two-impulse trajectory with a midcourse plane change is employed inside the shaded area. The result is the complete elimination of the region of high valued ΔV contours and the expansion of the area representing low-velocity requirements.

The results of calculations for transfers ending in the establishment of a satellite orbit around Mars at an altitude of 300 naut miles are shown in Fig. 6. Here again the intermediate impulse is employed only where it is advantageous. Since the magnitude of the impulse at point A to establish the satellite orbit is dependent upon the vehicle's velocity relative to Mars at point A, it is expected that the optimum location for point I generally will not be the same as that of the probe. This is verified by the analysis in Appendix B and by the characteristics of the curves of Fig. 6 which are different from the corresponding curves for the probe in Fig. 5.

Since the propellant for the intermediate impulse must be stored for a long period of time in space, the magnitude of this impulse is of great practical importance in the case of the probe. Of course, for the planetary orbital case a large amount of propellant must be stored anyway for the final impulse at point A. Corresponding to the period of high single-impulse velocity requirements in Fig. 4 for a 300-day trip time, the magnitude of the intermediate impulse is plotted against launch date in Fig. 7 as a fractional part of the total impulse required for the Mars probe. The maximum magnitude is approximately 13% of the total. Similar results are obtained for other trip times.

Conclusion

When an intermediate impulse is employed where advantageous to eliminate the periods of ΔV requirements which are due to the planetary orbit inclinations, the resulting total ΔV requirements correspond very closely to those for the case of coplanar planetary orbits. There are examples in the coplanar case which require the use of 180° or near 180° trajectories, e.g., the Hohmann transfer between coplanar circular orbits. Also the timing of nonstop, round-trip, interplanetary reconnaissance missions may require a near 180° segment. With the use of the optimum midcourse plane change, the advantages of these coplanar maneuvers can be realized in three dimensions.

Appendix A: Analytical Determination of Minimum ΔV Conditions for the Probe

Referring once more to the geometry and nomenclature of Fig. 2, it is seen that the position of the intermediate impulse point is defined by the intersection of three surfaces: 1) a plane perpendicular to plane 1 and forming an angle of θ_I with r_D; 2) plane 2 defined by the dihedral angle ϕ_2; and 3) the cylindrical surface defined by the elements of the two-dimensional transfer trajectory. The mathematical problem is to determine for the condition of minimum total ΔV the relationships among the intermediate impulse point coordinates ϕ_2 and θ_I, the configuration variables that are r_1, r_A, θ_A, θ_N, ϕ_4, and the elements of the two-dimensional ballistic transfer conic, l, a, and γ.

To find these relationships, the general expression for ΔV is first derived, and then the standard technique of differentiation with respect to the independent variables is employed. An orthogonal Cartesian coordinate system shown in Fig. 2 with unit vectors **i**, **j**, and **k** is employed throughout the analysis. The vector **i** is in the direction of the radius from the sun to point D, **j** lies in plane 1 perpendicular to **i**, and **k** is perpendicular to plane 1. The velocity vector at any point on the planetary-intercept trajectory is taken to have **i** and **j** components equal to the **i** and **j** velocity components at the projected point on the two-dimensional trajectory plus a **k** component of the proper magnitude so that the total vector lies in one of the trajectory planes 2 or 3.

The velocity of the probe in the heliocentric coordinate system at the point of departure is

$$\mathbf{V}_{2D} = \rho_D \mathbf{i} + \nu_D \mathbf{j} + \nu_D \tan\phi_2 \mathbf{k} \quad \text{[A1]}$$

whereas the velocity with respect to the departure planet is

$$\mathbf{V}_{\infty D} = (\rho_D - \rho_1)\mathbf{i} + (\nu_D - \nu_1)\mathbf{j} + \nu_D \tan\phi_2 \mathbf{k} \quad \text{[A2]}$$

In order to achieve the velocity of Eq. [A2] at infinity, the necessary increase in velocity from that of a circular orbit around the departure planet is

$$\Delta V_D = (Z^2 + \nu_D{}^2 \tan^2\phi_2)^{1/2} - V_{0D} \quad \text{[A3]}$$

where

$$Z^2 \equiv 2V_{0D}{}^2 + (\rho_D - \rho_1)^2 + (\nu_D - \nu_1)^2$$

Eq. [A3] can be expanded in a McClaurin series in ϕ_2, which, after omitting terms of higher order than $\phi_2{}^2$, becomes

$$\Delta V_D = Z - V_{0D} + (\nu_D{}^2\phi_2{}^2/2Z) \quad \text{[A4]}$$

It can be shown easily that the velocity of the probe in plane 2 with respect to the heliocentric frame at point I must be

$$\mathbf{V}_{2I} = (\rho_I \cos\theta_I - \nu_I \sin\theta_I)\mathbf{i} + (\rho_I \sin\theta_I + \nu_I \cos\theta_I)\mathbf{j} + (\rho_I \sin\theta_I + \nu_I \cos\theta_I) \tan\phi_2 \mathbf{k} \quad \text{[A5]}$$

The intermediate impulse must be of proper magnitude to insure that the probe traverses the arrival planet (point A); i.e., plane 3 must contain point A. As an aid in the vector

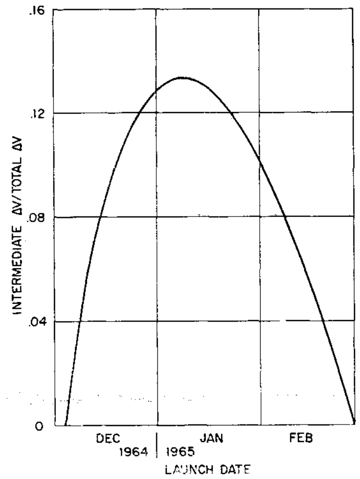

Fig. 7 Intermediate impulse as fraction of total impulse; trip time = 300 days

analysis, a normal vector to plane 3 is defined by the relation

$$\mathbf{N}_3 \equiv (\mathbf{r}_I \times \mathbf{r}_A/r_I r_A) = [\sin\theta_I \sin(\theta_A - \theta_N)\tan\phi_4 - \sin\theta_I \sin\theta_A \tan\phi_2]\mathbf{i} + [\sin\theta_I \cos\theta_A \tan\phi_2 - \cos\theta_I \sin(\theta_A - \theta_N)\tan\phi_4]\mathbf{j} + \sin(\theta_A - \theta_I)\mathbf{k} \quad \text{[A6]}$$

After the application of the intermediate impulse, in order for the velocity vector to lie in plane 3, $\mathbf{V}_{3I}\cdot\mathbf{N}_3 = 0$, and therefore

$$\mathbf{V}_{3I} = (\rho_I \cos\theta_I - \nu_I \sin\theta_I)\mathbf{i} + (\rho_I \sin\theta_I + \nu_I \cos\theta_I)\mathbf{j} + \left\{[\rho_I \sin\theta_I + \nu_I \cos\theta_I]\tan\phi_2 + \frac{\nu_I[\sin(\theta_A - \theta_N)\tan\phi_4 - \sin\theta_A \tan\phi_2]}{\sin(\theta_A - \theta_I)}\right\}\mathbf{k} \quad \text{[A7]}$$

The necessary change in velocity at I is

$$\Delta\mathbf{V}_I = \mathbf{V}_{3I} - \mathbf{V}_{2I} = \frac{\nu_I[\sin(\theta_A - \theta_N)\tan\phi_4 - \sin\theta_A \tan\phi_2]}{\sin(\theta_A - \theta_I)}\mathbf{k} \quad \text{[A8]}$$

The total velocity change can now be obtained by adding Eqs. [A4] and [A8]:

$$\Delta V = Z - V_{0D} + \frac{\nu_D^2}{2Z}\phi_2^2 + \left|\frac{\nu_I[\sin(\theta_A - \theta_N)\tan\phi_4 - \sin\theta_A \tan\phi_2]}{\sin(\theta_A - \theta_I)}\right| \quad \text{[A9]}$$

For stationary ΔV, $\partial\Delta V/\partial\phi_2 = 0$ and $\partial\Delta V/\partial\theta_I = 0$. Setting $\sec^2\phi_2 = 1 + \phi_2^2$, the derivative equation in ϕ_2 becomes

$$\pm\frac{\nu_I \sin\theta_A}{\sin(\theta_A - \theta_I)}\phi_2^2 + \frac{\nu_D^2}{Z}\phi_2 \pm \frac{\nu_I \sin\theta_A}{\sin(\theta_A - \theta_I)} = 0 \quad \text{[A10]}$$

which has a solution

$$\phi_2 = \pm\psi \pm (\psi^2 - 1)^{1/2} \quad \text{[A11]}$$

where

$$\psi \equiv \frac{\nu_D^2 \sin(\theta_A - \theta_I)}{2Z\nu_I \sin\theta_A} \quad \text{[A12]}$$

The sign choices in Eq. [A11] can be made by the following geometrical reasoning. The target point is above or below plane 1 according to the sign of $\sin(\theta_A - \theta_N)$; ϕ_2 must be positive if the target point is above plane 1 [i.e., $\sin(\theta_A - \theta_N) > 0$] and the total central angle of the transfer conic is less than π (i.e., $\sin\theta_A > 0$). If either one of these conditions is reversed, ϕ_2 must be negative, and of course if they are both reversed ϕ_2 will remain positive.

The second derivative equation with respect to θ_I provides the result that

$$\sin(\theta_A - \theta_I)(\partial\nu_I/\partial\theta_I) + \nu_I \cos(\theta_A - \theta_I) = 0 \quad \text{[A13]}$$

The circumferential velocity ν_I can be obtained from the general equation for a conic

$$r_I = l/(1 + m\cos\theta_I + q\sin\theta_I) \quad \text{[A14]}$$

where m and q are functions of the elements:

$$m = (l/r_D) - 1$$
$$q = -\sin\gamma = \pm\{[1 - (l/a)] - [1 - (l/r_D)]^2\}^{1/2} \quad \text{[A15]}$$

The sign of q must be chosen opposite to the sign of $\sin\gamma$, γ being the angle between the periapsis and the departure radius r_D. Then

$$\nu_I = (\mu l)^{1/2}/r_I = (\mu/l)^{1/2}(1 + m\cos\theta_I + q\sin\theta_I) \quad \text{[A16]}$$

and

$$\partial\nu_I/\partial\theta_I = (\mu/l)^{1/2}(-m\sin\theta_I + q\cos\theta_I) \quad \text{[A17]}$$

Substituting Eqs. [A16] and [A17] into Eq. [A13] and solving for θ_I, one obtains

$$\theta_I = \theta_A - \cos^{-1}(-m\cos\theta_A + q\sin\theta_A) \quad \text{[A18]}$$

Eqs. [A11] and [A18] give values for ϕ_2 and θ_I which produce stationary values of ΔV in Eq. [A9]. In order for a relative minimum to occur in ΔV, the conditions $B^2 - AC < 0$ and $A > 0$ must be satisfied, where $A \equiv \partial^2\Delta V/\partial\phi_2^2$; $B \equiv \partial^2\Delta V/\partial\phi_2\partial\theta_I$; and $C \equiv \partial^2\Delta V/\partial\theta_I^2$. A, B, and C are evaluated at $\phi_{2\text{sta}}$ and $\theta_{I\text{sta}}$.

Appendix B: Determination of Minimum ΔV Conditions for the Satellite

The total ΔV requirement for the satellite is the same as that for the probe with the addition of the impulse required to establish the satellite orbit at the arrival planet:

$$\Delta V = Z - V_{0D} + \frac{\nu_D^2\phi_2^2}{2Z} + \left|\frac{\nu_I[\sin(\theta_A - \theta_N)\tan\phi_4 - \sin\theta_A \tan\phi_2]}{\sin(\theta_A - \theta_I)}\right| + \Delta V_A \quad \text{[B1]}$$

The final impulse is equal to the difference between the peri-velocity of the hyperbola and circular velocity at the prescribed radius of the required circular satellite orbit at the arrival planet. Thus

$$\Delta V_A = (2V_{0A}^2 + V_{\infty A}^2)^{1/2} - V_{0A} \quad \text{[B2]}$$

where

$$V_{\infty A}^2 = |\mathbf{V}_{3A} - \mathbf{V}_{4A}|^2$$

Using the same coordinate system in Fig. 2, it can be shown that

$$\mathbf{V}_{3A} = (\rho_A \cos\theta_A - \nu_A \sin\theta_A)\mathbf{i} + (\rho_A \sin\theta_A + \nu_A \cos\theta_A)\mathbf{j} + \{[\rho_A + \nu_A \cot(\theta_A - \theta_I)]\sin(\theta_A - \theta_N)\tan\phi_4 - [\nu_A \sin\theta_I \tan\phi_2/\sin(\theta_A - \theta_I)]\}\mathbf{k} \quad \text{[B3]}$$

and

$$\mathbf{V}_{4A} = (\rho_4 \cos\theta_A - \nu_4 \sin\theta_A)\mathbf{i} + (\rho_4 \sin\theta_A + \nu_4 \cos\theta_A)\mathbf{j} + [\rho_4 \sin(\theta_A - \theta_N) + \nu_4 \cos(\theta_A - \theta_N)]\tan\phi_4\mathbf{k} \quad \text{[B4]}$$

Therefore

$$V_{\infty A}^2 = (\rho_A - \rho_4)^2 + (\nu_A - \nu_4)^2 + Y^2 \quad \text{[B5]}$$

where

$$Y \equiv [\rho_A + \nu_A \cot(\theta_A - \theta_I)]\sin(\theta_A - \theta_N)\tan\phi_4 - [\nu_A \sin\theta_I \tan\phi_2/\sin(\theta_A - \theta_I)] - [\rho_4 \sin(\theta_A - \theta_N) + \nu_4 \cos(\theta_A - \theta_N)]\tan\phi_4 \quad \text{[B6]}$$

If

$$X^2 \equiv 2V_{0A}^2 + (\rho_A - \rho_4)^2 + (\nu_A - \nu_4)^2 \quad \text{[B7]}$$

then by substitution Eq. [B2] becomes

$$\Delta V_A = (X^2 + Y^2)^{1/2} - V_{0A} \quad \text{[B8]}$$

Eq. [B8] is now expanded in a Taylor series in two independent variables (ϕ_2 and θ_I) about the point ϕ_2^*, θ_I^*, where ϕ_2^* is set somewhat arbitrarily at zero and θ_I^* is given the value of $\theta_{I\text{sta}}$ in the case of the probe. Terms in the series containing higher orders than ϕ_2^2 are not retained:

$$\Delta V_A = \Delta V_A^* + \sum_{n=1}^{\infty} \frac{1}{n!}\left[(\phi_2 - \phi_2^*)\frac{\partial}{\partial\phi_2} + (\theta_I - \theta_I^*)\frac{\partial}{\partial\theta_I}\right]^n_{(\phi_2^*,\theta_I^*)} \Delta V_A(\phi_2, \theta_I) \quad \text{[B9]}$$

After the indicated operations of Eq. [B9] are performed and the results are substituted into Eq. [B1], the total required velocity increase is

$$\Delta V = Z - V_{0D} + \frac{\nu_D^2\phi_2^2}{2Z} + \left|\frac{\nu_I[\sin(\theta_A - \theta_N)\tan\phi_4 - \sin\theta_A \tan\phi_2]}{\sin(\theta_A - \theta_I)}\right| + K_1 + K_2\phi_2^2 + K_3\phi_2 + K_4\phi_2\theta_I + K_5\theta_I + K_6\theta_I^2 \quad \text{[B10]}$$

where the constants denoted by the K's are

$$K_1 \equiv (X^2 + Y^{*2})^{1/2} - V_{0A} - \frac{Y^*}{(X^2 + Y^{*2})^{1/2}} \nu_A \csc^2(\theta_A - \theta_I^*) \sin(\theta_A - \theta_N) \tan\phi_4\theta_I^* + \frac{1}{2}\frac{X^2\theta_I^{*2}}{(X^2 + Y^{*2})^{3/2}} \times [2Y^*\csc^2(\theta_A - \theta_I^*) \cot(\theta_A - \theta_I^*) \sin(\theta_A - \theta_N) \times \tan\phi_4 + \nu_A^2 \csc^2(\theta_A - \theta_I^*) \sin^2(\theta_A - \theta_N) \tan^2\phi_4] \quad \text{[B11]}$$

$$K_2 \equiv \frac{1}{2}\frac{X^2}{(X^2 + Y^{*2})^{3/2}} \frac{\nu_A^2 \sin^2\theta_I^*}{\sin^2(\theta_A - \theta_I^*)} \quad \text{[B12]}$$

$$K_3 \equiv -\frac{Y^*}{(X^2 + Y^{*2})^{1/2}} \frac{\nu_A \sin\theta_I^*}{\sin(\theta_A - \theta_I^*)} + \frac{X^2\theta_I^*}{(X^2 + Y^{*2})^{3/2}} \times \left[\frac{Y^*\nu_A \sin\theta_A}{\sin^2(\theta_A - \theta_I^*)} + \frac{2\nu_A^2 \sin\theta_I^*}{\sin^3(\theta_A - \theta_I^*)} \cot(\theta_A - \theta_I^*) \times \sin(\theta_A - \theta_N) \tan\phi_4\right] \quad \text{[B13]}$$

$$K_4 \equiv -\frac{X^2}{(X^2 + Y^{*2})^{3/2}}\left[\frac{Y^*\nu_A \sin\theta_A}{\sin^2(\theta_A - \theta_I^*)} + \frac{2\nu_A^2 \sin\theta_I^*}{\sin^3(\theta_A - \theta_I^*)} \times \cot(\theta_A - \theta_I^*) \sin(\theta_A - \theta_N) \tan\phi_4\right] \quad \text{[B14]}$$

$$K_5 \equiv \frac{Y^*}{(X^2 + Y^{*2})^{1/2}} \nu_A \csc^2(\theta_A - \theta_I^*) \sin(\theta_A - \theta_N) \tan\phi_4 \quad \text{[B15]}$$

$$K_6 \equiv \frac{1}{2}\frac{X^2}{(X^2 + Y^{*2})^{3/2}}\left[2Y^* \csc^2(\theta_A - \theta_I^*) \sin(\theta_A - \theta_N) \tan\phi_4 + \nu_A^2 \csc^2(\theta_A - \theta_I^*) \sin^2(\theta_A - \theta_N) \tan^2\phi_4\right] \quad \text{[B16]}$$

For stationary ΔV, $\partial\Delta V/\partial\phi_2 = 0$ and $\partial\Delta V/\partial\theta_I = 0$. Employing the small angle approximation that $\tan\phi_2 = \phi_2$, the differentiation of Eq. [B10] with respect to ϕ_2 gives

$$\phi_2\left(\frac{\nu_D^2}{Z} + 2K_2\right) + K_3 + K_4\theta_I \pm \frac{\nu_I \sin\theta_A}{\sin(\theta_A - \theta_I)} = 0 \quad \text{[B17]}$$

and solving for ϕ_2

$$\phi_{2\text{sta}} = \frac{\pm[\nu_I \sin\theta_A/\sin(\theta_A - \theta_I)] - K_3 - K_4\theta_I}{(\nu_D^2/Z) + 2K_2} \quad \text{[B18]}$$

The choice of sign in Eqs. [B17] and [B18] is taken according to the same convention used for the probe.

Differentiation of Eq. [B10] with respect to θ_I gives

$$\frac{\partial\Delta V}{\partial\theta_I} = \left[\sin(\theta_A - \theta_N) \tan\phi_4 - \sin\theta_A \tan\phi_2\right] \times \frac{\sin(\theta_A - \theta_I)\,(\partial\nu_I/\partial\theta_I) + \nu_I \cos(\theta_A - \theta_I)}{\sin^2(\theta_A - \theta_I)} + K_4\phi_2 + K_5 + 2K_6\theta_I = 0 \quad \text{[B19]}$$

Since in the regions of interest ϕ_2 and ϕ_4 will always be small angles, a final approximation can be justified here. If all terms containing ϕ_2^2, ϕ_4^2, and $\phi_2\phi_4$ are neglected, $K_4\phi_2 = 0$, $K_5 = 0$, and $K_6 = 0$, and Eq. [B19] becomes identical with Eq. [A13] in the probe analysis. Therefore, to a first-order approximation, the $\theta_{I\text{sta}}$ for the satellite is identical to that for the probe.

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References

1 Battin, R. H., "The determination of round-trip planetary reconnaissance trajectories," J. Aero-Space Sci. **26**, 545–567 (1959).

2 Breakwell, J. V., Gillespie, R. W., and Ross, S., "Researches in interplanetary transfer," ARS J. **31**, 201–208 (1961).

3 London, H. S., Edelbaum, T. N., Fimple, W. R., and Gobetz, F. W., "Mission capabilities of ion engines using SNAP-8 power supplies," United Aircraft Corp. Research Labs. Rept. R-1383-3 (August 1961).

4 Bossart, K. J., "Departure and return in interplanetary flight," Aerospace Eng. **17**, 44–52 (October 1958).

5 Ruppe, H. O., "Interplanetary flight," *Handbook of Astronautical Engineering*, edited by H. H. Koelle (McGraw-Hill Book Co. Inc., New York, 1961), Chap. 9.32.